Reprint ISSN 0974-1518

INTERNATIONAL JOURNAL OF ENGINEERING RESEARCH AND INDUSTRIAL APPLICATIONS

(IJERIA)

www.ascent-journals.com

International J. of Engg. and Research of Indu. Appls. (IJERIA)
ISSN 0974-1518, Vol. 14, No. I (June 2025), pp. 1- 20

APPLICATIONS OF QUADRUPLED FIXED POINT THEOREMS IN C*− ALGEBRA VALUED GENERALIZED METRIC SPACES

RAJESH SHRIVASTAVA AND SACHIN MOKHLE

Abstract. We introduce the concept of quadrupled fixed point theorems in C^*−algebra-valued b-metric space and gives some basic fixed point theorems for self-map with contractive condition on such spaces. As applications, existence and uniqueness results for a type of operator equation and an integral equation are given.

1. Introduction

Since the year 1922, Banachs contraction principle, due to its simplicity and applicability, has became a very popular tool in modern analysis, especially in nonlinear analysis including its applications to differential and integral equations, variational inequality theory, complementarity problems, equilibrium problems, minimization problems and many others. Also, many authors have improved, extended and generalized this contraction principle in several ways. Existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [20] further studied by Nieto and Rodriguez - Lopez [17]. Samet and Vetro [21] introduced the notion of fixed point of N order in case of single-valued mappings. It should be noted that through the coupled fixed point (for N = 2) and tripled fixed point (for N = 3) technique we cannot solve a system with the following form:

$$\begin{aligned} x^4+6yzw-9x+12 &= 0, \\ y^4+6xzw-9y+12 &= 0, \\ z^4+6yxw-9z+12 &= 0 \\ w^4+6yxz-9w+12 &= 0. \end{aligned}$$

In particular for $N = 4$ (Quadruple case) i.e., Let $(X, \preceq)$ be partially ordered set and (X, d) be a complete metric space. We consider the following partial order on the product space $X^4 = X \times X \times X \times X$

$$(u, v, r, t) \preceq (x, y, z, w) \;\; iff \;\; x \preceq u, \;\; y \preceq \;\; v, \;\; z \preceq r, \;\; t \preceq w, \tag{1.1}$$

2000 *Mathematics Subject Classification.* 47H10; 46L07.

Key words and phrases. C^*−algebra, C^*−algebra-valued b-metric, contractive mapping, quadrupled fixed point theorem.

where $(u, v, r, t), (x, y, z, w) \in X^4$.

In 1989, Bakhtin [4] introduced b-metric space as a generalization of metric space. Since then, more other generalized b-metric spaces such as b-metric-like spaces [2], quasi-b-metric spaces [22] and quasi-b-metric-like spaces [23] were introduced. Recently, Ma and Jiang [15] initially introduced the concept of a C^* -algebra-valuedb-metric space which generalized the concept of b-metric spaces, and they established certain basic fixed point theorems for self-map with contractive condition in this new setting. In 2016, Kamran et al. [9] also introduced the concept of C^* -algebra-valued b-metric space, and they generalized the Banach contraction principle on such spaces.

Regarding this partial order Karapinar [11] give the following definitions,

Definition 1. *Let $(X, \preceq)$ be partially ordered set and $F : X^4 \to X$. We say that F has the mixed monotone property if $F(x, y, z, w)$ is monotone non decreasing in x and z and it is monotone non increasing in y and w, that is, for any $x, y, z, w \in X$*

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 &\preceq x_2 \Longrightarrow F(x_1, y, z, w) \preceq F(x_2, y, z, w) \\ y_1, y_2 \in X, \quad y_1 &\preceq y_2 \Longrightarrow F(x, y_2, z, w) \preceq F(x, y_1, z, w) \\ z_1, z_2 \in X, \quad z_1 &\preceq z_2 \Longrightarrow F(x, y, z_1, w) \preceq F(x, y, z_2, w) \\ w_1, w_2 \in X, \quad w_1 &\preceq w_2 \Longrightarrow F(x, y, z, w_2) \preceq F(x, y, z, w_1) \qquad (1.2) \end{aligned}$$

Definition 2. *An element $(x, y, z, w) \in X^4$ is called a quadruple fixed point of $F : X^4 \to X$ if*

$$\begin{aligned} F(x, y, z, w) &= x, \quad F(y, z, w, x) = y, \\ F(z, w, x, y) &= z, \quad F(w, x, y, z) = w. \qquad (1.3) \end{aligned}$$

Definition 3. *Let (X, d) be a complete metric space. It is called metric on X^4, the mapping $d : X \times X \to X$ with*

$$d[(x, y, z, t), (u, v, w, s)] = d(x, u) + d(y, v) + d(z, w) + d(t, s).$$

In this paper we studied some quadrupled fixed point theorems in the context of complete C^* -algebra-valued metric spaces.

Motivated by the work in [15, 9, 11, 12, 10], in this paper, we will establish quadrupled fixed point theorems in C^* -algebra-valued b-metric space. More precisely, we will prove some quadrupled fixed point theorems for the mapping with different contractive conditions on such spaces.

For convenience, we now recall some basic definitions, notations, and results of C^* -algebra. The details of C^* -algebras can be found in [13].

Let $\mathbb{A}$ be an algebra. An involution on $\mathbb{A}$ is a conjugate linear map $a \mapsto a^*$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a*-algebra. If $\mathbb{A}$ contains the identity element $1_{\mathbb{A}}$, then $(\mathbb{A}, *)$ is called a unital*-algebra. A *-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ is said to be a Banach *-algebra. Moreover, if for all $a \in \mathbb{A}$, we have $\|a^*a\| = \|a\|^2$ in a Banach *-algebra, then $\mathbb{A}$ is known as a C^* -algebra. An element a of a C^* -algebra $\mathbb{A}$ is positive if $a = a^*$ and its spectrum $\sigma(a) \subset \mathbb{R}_+$, where $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda 1_{\mathbb{A}}\text{-}a \text{ is not invertible}\}$. Each positive element a of C^* -algebra $\mathbb{A}$ has a unique positive square root. The set of all positive elements will be denoted by $\mathbb{A}_+$. There is a natural partial ordering on the elements of $\mathbb{A}$ given by

$$a \preceq b \iff b - a \in \mathbb{A}_+.$$

If $a \in \mathbb{A}_+$, then we write $a \succeq 0_{\mathbb{A}}$, where $0_{\mathbb{A}}$ is the zero element of$\mathbb{A}$. In the following, we always assume that $\mathbb{A}$ is a unital C^* -algebra with identity element$1_{\mathbb{A}}$.

Let $\mathbb{A}' = \{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$, and $\mathbb{A}'_+ = \mathbb{A}_+ \cap \mathbb{A}'$. From [15, 9], we now give the definition of C^* -algebra-valued b-metric as follows.

Definition 4. *Let $\mathbb{A}$ be a C^* -algebra, and X be a nonempty set. Let $b \in \mathbb{A}'_+$ be such that $\|b\| \geq 1$. A mapping $d_b : X \times X \to \mathbb{A}_+$ is said to be a C^* -algebra-valued b-metric on X if the following conditions hold for all $x, y, z \in \mathbb{A}$:*

1. *$d_b(x, y) = 0_{\mathbb{A}}$ if and only if $x = y$;*
2. *$d_b(x, y) = d_b(y, x)$;*
3. *$d_b(x, y) \preceq b[d_b(x, z) + d_b(z, y)]$.*

The triplet $(X, \mathbb{A}, d_b)$ is called a C^* -algebra-valued b-metric space with coefficient b.

Remark 5. *From Example 2.1 in [9], we know that a C^* -algebra-valued metric space is C^* -algebra-valued b-metric space, but the converse is not true.*

Definition 6. *Let $(X, \mathbb{A}, d_b)$ be a C^* -algebra-valued b-metric space, $x \in X$, and $\{x_n\}$ a sequence in X. Then:*

1. *$\{x_n\}$ converges to x with respect to $\mathbb{A}$ whenever for any $\varepsilon > 0$ there is an$N \in \mathbb{N}$ such that $\|d_b(x_n, x)\| < \varepsilon$ for all $n > N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.*
2. *$\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$ if for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $\|d_b(x_n, x_m)\| < \varepsilon$ for all $n, m > N$.*
3. *$(X, \mathbb{A}, d_b)$ is complete if every Cauchy sequence in X is convergent with respect to $\mathbb{A}$.*

Lemma 7. *Assume that $\mathbb{A}$ is a unital C^* -algebra with a unit $1_{\mathbb{A}}$.*

1) *For any $x \in \mathbb{A}_+$, we have $x \preceq 1_{\mathbb{A}} \Leftrightarrow \|x\| \leq 1$;*
2) *if $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $1_{\mathbb{A}} - a$ is invertible and $\|a(1_{\mathbb{A}} - a)^{-1}\| < 1$;*
3) *assume that $a, b \in \mathbb{A}$ with $a, b \succeq 0_{\mathbb{A}}$ and $ab = ba$, then $ab \succeq 0_{\mathbb{A}}$;*
4) *let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq 0_{\mathbb{A}}$,and $1_{\mathbb{A}} - a \in \mathbb{A}'_+$ is an invertible operator, then*

$$(1_{\mathbb{A}} - a)^{-1}b \succeq (1_{\mathbb{A}} - a)^{-1}c;$$

5) *if* $b, c \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ *and* $a \in \mathbb{A}$, *then* $b \preceq c \Longrightarrow a^*ba \preceq a^*ca$;

6) *if* $0_{\mathbb{A}} \preceq a \preceq b$, *then* $\|a\| \leq \|b\|$.

Lemma 8. *The sum of two positive elements in a* C^* *-algebra is a positive element.*

Remark 9. *From Lemma's 7(3) and 8 , we know that the condition* $b \in \mathbb{A}'_+$ *in Definition 4 is necessary, in this case, we see that* $b[d_b(x,z) + d_b(z,y)]$ *is a positive element.*

2. Main Results

From now on, we denote $X^4 = X \times X \times X \times X$. We begin this section at giving a new quadrupled fixed point theorem in the setting of C^* -algebra-valued b-metric space.

Theorem 10. *Let* $(X, \mathbb{A}, d_b)$ *be a complete* C^* *-valued b-metric space. Assume that the mapping* $T : X^4 \to X$ *satisfies the following condition:*

$$d_b\big(T(x,y,z,t), T(u,v,w,s)\big) \preceq a^*d_b(x,u)a + a^*d_b(y,v)a + a^*d_b(z,w)a + a^*d_b(t,s)a, \tag{2.1}$$

for all $x, y, z, t, u, v, w, s \in X$, *where* $a \in \mathbb{A}$ *with* $4\|a\|^2\|b\|^2 < 1$. *Then* T *has a unique quadrupled fixed point in* X. *Moreover,* T *has a unique fixed point in* X.

Proof. Let $x_0, y_0, z_0, t_0 \in X$. Define four sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ in X by the iterative scheme as

$$x_{n+1} = T(x_n, y_n, z_n, t_n), \qquad y_{n+1} = T(y_n, z_n, t_n, x_n),$$

$$z_{n+1} = T(z_n, t_n, x_n, y_n), \qquad t_{n+1} = T(t_n, x_n, y_n, z_n).$$

By using the condition 2.1, for $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d_b(x_n, x_{n+1}) =& d_b\big(T(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), T(x_n, y_n, z_n, t_n)\big) \\ \preceq& a^*d_b(x_{n-1}, x_n)a + a^*d_b(y_{n-1}, y_n)a + a^*d_b(z_{n-1}, z_n)a + a^*d_b(t_{n-1}, t_n)a \\ =& a^*M_na, \end{aligned} \tag{2.2}$$

where

$$M_n = d_b(x_{n-1}, x_n) + d_b(y_{n-1}, y_n) + d_b(z_{n-1}, z_n) + d_b(t_{n-1}, t_n). \tag{2.3}$$

Similarly, we get

$$d_b(y_n, y_{n+1}) = d_b\big(T(y_{n-1}, z_{n-1}, t_{n-1}, x_{n-1}), T(y_n, z_n, t_n, x_n)\big) \preceq a^*M_na. \tag{2.4}$$

$$d_b(z_n, z_{n+1}) = d_b\big(T(z_{n-1}, t_{n-1}, x_{n-1}, y_{n-1}), T(z_n, t_n, x_n, y_n)\big) \preceq a^*M_na. \tag{2.5}$$

and

$$d_b(t_n, t_{n+1}) = d_b\big(T(t_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}), T(t_n, x_n, y_n, z_n)\big) \preceq a^*M_na. \tag{2.6}$$

By 2.2, 2.3, 2.4, 2.5 and 2.6 we have

$$\begin{aligned} M_{n+1} =& d_b(x_n,x_{n+1})+d_b(y_n,y_{n+1})+d_b(z_n,z_{n+1})+d_b(t_n,t_{n+1}) \\ \preceq& a^*\big[d_b(x_{n-1},x_n)+d_b(y_{n-1},y_n)+d_b(z_{n-1},z_n)+d_b(t_{n-1},t_n)\big]a \\ &+a^*\big[d_b(y_{n-1},y_n)+d_b(z_{n-1},z_n)+d_b(t_{n-1},t_n)+d_b(x_{n-1},x_n)\big]a \\ &+a^*\big[d_b(z_{n-1},z_n)+d_b(t_{n-1},t_n)+d_b(x_{n-1},x_n)+d_b(y_{n-1},y_n)\big]a \\ &+a^*\big[d_b(t_{n-1},t_n)+d_b(x_{n-1},x_n)+d_b(y_{n-1},y_n)+d_b(z_{n-1},z_n)\big]a \\ \preceq& (2a)^*\big[d_b(x_{n-1},x_n)+d_b(y_{n-1},y_n)+d_b(z_{n-1},z_n)+d_b(t_{n-1},t_n)\big](2a) \\ \preceq& (2a)^*M_n(2a). \end{aligned} \tag{2.7}$$

Thus, from 2.7 and Lemma 7 , we have

$$0_{\mathbb{A}} \preceq M_{n+1} \preceq (2a)^*M_n(2a) \preceq \cdots \preceq \big[(2a)^*\big]^n M_1(2a)^n. \tag{2.8}$$

If $M_1 = 0_{\mathbb{A}}$, then from Definition 3 we easily know that (x_0, y_0, z_0, t_0) is a quadrupled fixed point of the mapping T. Now, let $0_{\mathbb{A}} \preceq M_1$. Let $n, m \in \mathbb{N}$ with $m > n$, by using Definition 4, it follows that

$$\begin{aligned} d_b(x_n,x_m) \preceq& b\big[d_b(x_n,x_{n+1})+d_b(x_{n+1},x_m)\big] \\ \preceq& bd_b(x_n,x_{n+1})+b^2\big[d_b(x_{n+1},x_{n+2})+d_b(x_{n+2},x_m)\big] \\ =& bd_b(x_n,x_{n+1})+b^2d_b(x_{n+1},x_{n+2})+b^2d_b(x_{n+2},x_m) \\ \preceq& bd_b(x_n,x_{n+1})+b^2d_b(x_{n+1},x_{n+2})+\cdots \\ &+b^{m-n-1}d_b(x_{m-2},x_{m-1})+b^{m-n-1}d_b(x_{m-1},x_m). \end{aligned} \tag{2.9}$$

Similarly, we have

$$\begin{aligned} d_b(y_n,y_m) \preceq& bd_b(y_n,y_{n+1})+b^2d_b(y_{n+1},y_{n+2})+\cdots \\ &+b^{m-n-1}d_b(y_{m-2},y_{m-1})+b^{m-n-1}d_b(y_{m-1},y_m) \end{aligned} \tag{2.10}$$

$$\begin{aligned} d_b(z_n,z_m) \preceq& bd_b(z_n,z_{n+1})+b^2d_b(z_{n+1},z_{n+2})+\cdots \\ &+b^{m-n-1}d_b(z_{m-2},z_{m-1})+b^{m-n-1}d_b(z_{m-1},z_m). \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} d_b(t_n,t_m) \preceq& bd_b(t_n,t_{n+1})+b^2d_b(t_{n+1},t_{n+2})+\cdots \\ &+b^{m-n-1}d_b(t_{m-2},t_{m-1})+b^{m-n-1}d_b(t_{m-1},t_m). \end{aligned} \tag{2.12}$$

Hence,

$$
\begin{aligned}
&d_b(x_n,x_m)+d_b(y_n,y_m)+d_b(z_n,z_m)+d_b(t_n,t_m)\\
&\quad\preceq bM_{n+1}+b^2M_{n+2}+\cdots+b^{m-n-1}M_{m-1}+b^{m-n-1}M_m\\
&\quad\preceq b\big[(2a)^*\big]^nM_1(2a)^n+b^2\big[(2a)^*\big]^{n+1}M_1(2a)^{n+1}+\cdots\\
&\qquad+b^{m-n-1}\big[(2a)^*\big]^{m-2}M_1(2a)^{m-2}+b^{m-n-1}\big[(2a)^*\big]^{m-1}M_1(2a)^{m-1}
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
&=b\sum_{i=n}^{m-2}b^{i-n}\big[(2a)^*\big]^iM_1(2a)^i+b^{m-n-1}\big[(2a)^*\big]^{m-1}M_1(2a)^{m-1}\\
&=b\sum_{i=n}^{m-2}b^{i-n}\big[(2a)^*\big]^iM_1^{\frac12}M_1^{\frac12}(2a)^i+b^{m-n-1}\big[(2a)^*\big]^{m-1}M_1^{\frac12}M_1^{\frac12}(2a)^{m-1}\\
&=b\sum_{i=n}^{m-2}b^{i-n}\big(M_1^{\frac12}(2a)^i\big)^*\big(M_1^{\frac12}(2a)^i\big)+b^{m-n-1}\big(M_1^{\frac12}(2a)^{m-1}\big)^*\big(M_1^{\frac12}(2a)^{m-1}\big)\\
&=b\sum_{i=n}^{m-2}b^{i-n}\big|M_1^{\frac12}(2a)^i\big|^2+b^{m-n-1}\big|M_1^{\frac12}(2a)^{m-1}\big|^2\\
&\preceq\left\|b\sum_{i=n}^{m-2}b^{i-n}\big|M_1^{\frac12}(2a)^i\big|^2\right\|1_{\mathbb{A}}+\big\|b^{m-n-1}\big|M_1^{\frac12}(2a)^{m-1}\big|^2\big\|1_{\mathbb{A}}\\
&\preceq\|b\|\sum_{i=n}^{m-2}\big\|b^{i-n}\big\|\big\|M_1^{\frac12}\big\|^2\big\|(2a)^i\big\|^21_{\mathbb{A}}+\big\|b^{m-n-1}\big\|\big\|M_1^{\frac12}\big\|^2\big\|(2a)^{m-1}\big\|^21_{\mathbb{A}}\\
&\preceq\|b\|^{1-n}\big\|M_1^{\frac12}\big\|^2\sum_{i=n}^{m-2}\|b\|^i\big\|(2a)^2\big\|^i1_{\mathbb{A}}+\|b\|^{-n}\big\|M_1^{\frac12}\big\|^2\|b\|^{m-1}\big\|(2a)^2\big\|^{m-1}1_{\mathbb{A}}\\
&=\|b\|^{1-n}\big\|M_1^{\frac12}\big\|^2\sum_{i=n}^{m-2}\big(4\|a\|^2\|b\|\big)^i1_{\mathbb{A}}+\|b\|^{-n}\big\|M_1^{\frac12}\big\|^2\big(4\|a\|^2\|b\|^2\big)^{m-1}1_{\mathbb{A}}\\
&\to0_{\mathbb{A}}\quad(\text{as } m,n\to\infty),
\end{aligned}
\tag{2.14}
$$

by the condition $4\|a\|^2\|b\|^2<1$ and $\|b\|\geq1$. Hence $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are Cauchy sequences in X. By the completeness of $(X,\mathbb{A},d)$, there exist $x^*,y^*,z^*,t^*\in X$ such that $x_n\to x^*$, $y_n\to y^*$, $z_n\to z^*$ and $t_n\to t^*$ as $n\to\infty$. We now show that $T(x^*,y^*,z^*,t^*)=x^*$, $T(y^*,z^*,t^*,x^*)=y^*$, $T(z^*,t^*,x^*,y^*)=z^*$ and $T(t^*,x^*,y^*,z^*)=t^*$. From Definition 4 and 2.1, we get

$$
\begin{aligned}
0_{\mathbb{A}}\preceq&d_b\big(T\big(x^*,y^*,z^*,t^*\big),x^*\big)\preceq b\big[d_b\big(T\big(x^*,y^*,z^*,t^*\big),x_{n+1}\big)+d_b\big(x_{n+1},x^*\big)\big]\\
=&b\big[d_b\big(T\big(x^*,y^*,z^*,t^*\big),T(x_n,y_n,z_n,t_n)\big)+d_b\big(x_{n+1},x^*\big)\big]\\
\preceq&ba^*d_b\big(x^*,x_n\big)a+ba^*d_b\big(y^*,y_n\big)a+ba^*d_b\big(z^*,z_n\big)a\\
+&ba^*d_b\big(t^*,t_n\big)a+bd_b\big(x_{n+1},x^*\big)\to0_{\mathbb{A}}\quad(n\to\infty).
\end{aligned}
\tag{2.15}
$$

So, $T(x^*,y^*,z^*,t^*)=x^*$.

Similarly, we have $T(y^*, z^*, t^*, x^*) = y^*$, $T(z^*, t^*, x^*, y^*) = z^*$ and $T(t^*, x^*, y^*, z^*) = t^*$. Thus, (x^*, y^*, z^*, t^*) is a quadrupled fixed point of T.

If there exists another quadrupled fixed point (u, v, w, s) of T, then

$$\begin{aligned}
0_{\mathbb{A}} \preceq& d_b(x^*, u) \\
=& d_b(T(x^*, y^*, z^*, t^*), T(u, v, w, s)) \\
\preceq& a^* d_b(x^*, u) a + a^* d_b(y^*, v) a + a^* d_b(z^*, w) a + a^* d_b(t^*, s) a, \\
0_{\mathbb{A}} \preceq& d_b(y^*, v) \\
=& d_b(T(y^*, z^*, t^*, x^*), T(v, w, s, u)) \\
\preceq& a^* d_b(y^*, v) a + a^* d_b(z^*, w) a + a^* d_b(t^*, s) a + a^* d_b(x^*, u) a, \\
0_{\mathbb{A}} \preceq& d_b(z^*, w) \\
=& d_b(T(z^*, t^*, x^*, y^*), T(w, s, u, v)) \\
\preceq& a^* d_b(z^*, w) a + a^* d_b(t^*, s) a + a^* d_b(x^*, u) a + a^* d_b(y^*, v) a, \\
0_{\mathbb{A}} \preceq& d_b(t^*, s) \\
=& d_b(T(t^*, x^*, y^*, z^*), T(s, u, v, w)) \\
\preceq& a^* d_b(t^*, s) a + a^* d_b(x^*, u) a + a^* d_b(y^*, v) a + a^* d_b(z^*, w) a,
\end{aligned} \tag{2.16}$$

which implies that

$$\begin{aligned}
0_{\mathbb{A}} \preceq& d_b(x^*, u) + d_b(y^*, v) + d_b(z^*, w) + d_b(t^*, s) \\
\preceq& (2a)^* \big(d_b(x^*, u) + d_b(y^*, v) + d_b(z^*, w) + d_b(t^*, s)\big)(2a).
\end{aligned} \tag{2.17}$$

Thus, we have

$$\begin{aligned}
0 \leq& \|d_b(x^*, u) + d_b(y^*, v) + d_b(z^*, w) + d_b(t^*, s)\| \\
\leq& \|2a\|^2 \|d_b(x^*, u) + d_b(y^*, v) + d_b(z^*, w) + d_b(t^*, s)\| \\
<& \frac{1}{\|b\|^2} \|d_b(x^*, u) + d_b(y^*, v) + d_b(z^*, w) + d_b(t^*, s)\| \\
\leq& \|d_b(x^*, u) + d_b(y^*, v) + d_b(z^*, w) + d_b(t^*, s)\|,
\end{aligned} \tag{2.18}$$

which is a contradiction. Thus, $(u, v, w, s) = (x^*, y^*, z^*, t^*)$, that is, the quadrupled fixed point is unique. Finally, we will prove that T has a unique fixed point. Since

$$\begin{aligned}
0_{\mathbb{A}} \preceq& d_b(x^*, y^*) = d_b(T(x^*, y^*, z^*, t^*), T(y^*, z^*, t^*, x^*)) \\
\preceq& a^* d_b(x^*, y^*) a + a^* d_b(y^*, z^*) a + a^* d_b(z^*, t^*) a + a^* d_b(t^*, x^*) a \\
=& (2a)^* d_b(x^*, y^*)(2a),
\end{aligned} \tag{2.19}$$

we have

$$\|d_b(x^*, y^*)\| \leq 4\|a\|^2 \|d_b(x^*, y^*)\|. \tag{2.20}$$

It follows from the condition $4\|a\|^2 < \frac{1}{\|b\|^2} \leq 1$ that $\|d_b(x^*, y^*)\| = 0$. Hence, $x^* = y^* = z^* = t^*$. The proof is completed.

□

Theorem 11. *Let (X, A, d) be a complete C^*-algebra-valued metric space. Suppose that the mapping $F : X^4 \to X$ satisfies the following condition*

$$d(F(x, y, z, t), F(u, v, w, s)) \leq a^* d(x, u)a + a^* d(y, v)a + a^* d(z, w)a + a^* d(t, s)a,$$

for any $x, y, z, t, u, v, w, s \in X$, where $a \in A$ with $\|a\| < \frac{1}{2}$. Then F has a unique quadrupled fixed point. Moreover, F has a unique fixed point in X.

Proof. Taking $b = 1_{\mathbb{A}}$, becomes a special case of Theorem 10.

□

Theorem 12. *Let $(X, \mathbb{A}, d_b)$ be a complete C^* -valued b-metric space. Assume that the mapping $T : X^4 \to X$ satisfies the following condition:*

$$d_b\big(T(x, y, z, t), T(u, v, w, s)\big) \preceq a_1 d_b\big(T(x, y, z, t), u\big) + a_2 d_b\big(T(u, v, w, s), x\big), \tag{2.21}$$

where $x, y, z, t, u, v, w, s \in X$ and $a_1, a_2 \in \mathbb{A}'_+$ with $\|a_1 + a_2\|\|b\|^2 < 1$. Then T has a unique quadrupled fixed point in X. Moreover, T has a unique fixed point in X.

Proof. From $a_1, a_2 \in \mathbb{A}'_+$ and Lemma 8, we see that $a_1 d_b(T(x, y, z, t), u) + a_2 d_b(T(u, v, w, s), x)$ is a positive element. Choose $x_0, y_0, z_0, t_0 \in X$. Set $x_{n+1} = T(x_n, y_n, z_n, t_n)$,$y_{n+1} = T(y_n, z_n, t_n, x_n)$,$z_{n+1} = T(z_n, t_n, x_n, y_n)$ and $t_{n+1} = T(t_n, x_n, y_n, z_n)$ for $n = 0, 1, \ldots$. Applying 2.21, we have

$$\begin{aligned} d_b(x_n, x_{n+1}) =& d_b\big(T(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), T(x_n, y_n, z_n, t_n)\big) \\ \preceq& a_1 d_b\big(T(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), x_n\big) + a_2 d_b\big(T(x_n, y_n, z_n, t_n), x_{n-1}\big) \\ =& a_2 d_b(x_{n+1}, x_{n-1}) \preceq a_2 b\big[d_b(x_{n+1}, x_n) + d_b(x_n, x_{n-1})\big] \\ \preceq& a_2 b^2 d_b(x_{n+1}, x_n) + a_2 b d_b(x_n, x_{n-1}), \end{aligned} \tag{2.22}$$

which implies that

$$\big(1_{\mathbb{A}} - a_2 b^2\big) d_b(x_n, x_{n+1}) \preceq a_2 b d_b(x_n, x_{n-1}). \tag{2.23}$$

Moreover, we obtain

$$\begin{aligned} d_b(x_{n+1}, x_n) =& d_b\big(T(x_n, y_n, z_n, t_n), T(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1})\big) \\ \preceq& a_1 d_b\big(T(x_n, y_n, z_n, t_n), x_{n-1}\big) + a_2 d_b\big(T(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), x_n\big) \\ =& a_1 d_b(x_{n+1}, x_{n-1}) \preceq a_1 b^2 d_b(x_{n+1}, x_n) + a_1 b d_b(x_n, x_{n-1}), \end{aligned} \tag{2.24}$$

which yields

$$\big(1_{\mathbb{A}} - a_1 b^2\big) d_b(x_n, x_{n+1}) \preceq a_1 b d_b(x_n, x_{n-1}). \tag{2.25}$$

From 2.23 and 2.25, we get

$$\left(1_{\mathbb{A}} - \frac{(a_1+a_2)b^2}{2}\right)d_b(x_n, x_{n+1}) \preceq \frac{(a_1+a_2)b}{2}d_b(x_n, x_{n-1}). \tag{2.26}$$

Since $a_1, a_2, b \in \mathbb{A}'_+$, we have $\frac{(a_1+a_2)b}{3} \in \mathbb{A}'_+$ and $\frac{(a_1+a_2)b^2}{3} \in \mathbb{A}'_+$. Moreover, from the condition $\|(a_1+a_2)\|\|b\|^2 < 1$, we get

$$\left\|\frac{(a_1+a_2)b}{3}\right\| \leq \frac{1}{3}\left\|(a_1+a_2)\right\|\|b\| \leq \frac{1}{3}\left\|(a_1+a_2)\right\|\|b\|^2 < \frac{1}{3} \tag{2.27}$$

and

$$\left\|\frac{(a_1+a_2)b^2}{3}\right\| \leq \frac{1}{3}\left\|(a_1+a_2)\right\|\|b\|^2 < \frac{1}{3}, \tag{2.28}$$

which implies that $\left(1_{\mathbb{A}} - \frac{(a_1+a_2)b}{3}\right)^{-1} \in \mathbb{A}'_+$ and $\left(1_{\mathbb{A}} - \frac{(a_1+a_2)b^2}{3}\right)^{-1} \in \mathbb{A}'_+$ with

$$\left\|\left(1_{\mathbb{A}} - \frac{(a_1+a_2)b^2}{3}\right)^{-1}\frac{(a_1+a_2)b^2}{3}\right\| < 1 \tag{2.29}$$

by Lemma 7 (2). Thus, we have by 2.26

$$d_b(x_{n+1}, x_n) \preceq h d_b(x_n, x_{n-1}),$$

where

$$h = \left(1_{\mathbb{A}} - \frac{(a_1+a_2)b^2}{3}\right)^{-1}\frac{(a_1+a_2)b}{3} \tag{2.30}$$

with $\|h\| \leq \|hb\| < 1$ by 2.29. Inductively, for all $n \in \mathbb{N}$, we have

$$d_b(x_{n+1}, x_n) \preceq h^n d_b(x_1, x_0) = h^n m_0, \tag{2.31}$$

where $m_0 = d_b(x_1, x_0)$. Let $n, m \in \mathbb{N}$ with $m > n$, by using Definition 4 and 2.29-2.31, we have

$$\begin{aligned} d_b(x_n, x_m) \preceq & b\big[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_m)\big] \\ \preceq & bd_b(x_n, x_{n+1}) + b^2\big[d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_m)\big] \\ \preceq & bd_b(x_n, x_{n+1}) + b^2 d_b(x_{n+1}, x_{n+2}) + \cdots \\ & + b^{m-n-1}\big[d_b(x_{m-2}, x_{m-1}) + d_b(x_{m-1}, x_m)\big] \\ \preceq & bh^n m_0 + b^2 h^{n+1} m_0 + \cdots + b^{m-n-1} h^{m-2} m_0 + b^{m-n-1} h^{m-1} m_0. \\ = & \sum_{i=1}^{m-n-1} b^i h^{n+i-1} m_0 + b^{m-n-1} h^{m-1} m_0 \\ = & \sum_{i=1}^{m-n-1} \big|m_0^{\frac{1}{2}} h^{\frac{n+i-1}{2}} b^{\frac{i}{2}}\big|^2 + \big|m_0^{\frac{1}{2}} h^{\frac{m-1}{2}} b^{\frac{m-n-1}{2}}\big|^2 \\ \preceq & \|m_0\| \sum_{i=1}^{m-n-1} \|h\|^{n-1} \|hb\|^i 1_{\mathbb{A}} + \|m_0\| \|h\|^n \|hb\|^{m-n-1} 1_{\mathbb{A}} \\ \preceq & \frac{\|m_0\| \|h\|^{n-1} \|hb\|}{1 - \|hb\|} 1_{\mathbb{A}} + \|M_0\| \|h\|^n \|hb\|^{m-n-1} 1_{\mathbb{A}} \\ \to & 0_{\mathbb{A}} \quad (\text{as } m, n \to \infty). \end{aligned} \tag{2.32}$$

Hence $\{x_n\}$ is a Cauchy sequence in X. Similarly, we can prove that $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are also a Cauchy sequence in X. Since $(X, \mathbb{A}, d_b)$ is complete, we see that $\{x_n\}$ $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ converge to some $u \in X$, $v \in X$, $w \in X$ and $s \in X$ respectively. In the following, we will show that $T(u, v, w, s) = u$, $T(v, w, s, u) = v$, $T(w, s, u, v) = w$ and $T(s, u, v, w) = s$. By 2.21, we get

$$\begin{aligned} d_b\big(T(u, v, w, s), u\big) \preceq & b\big[d_b\big(x_{n+1}, T(u, v, w, s)\big) + d_b(x_{n+1}, u)\big] \\ \preceq & b\big[d_b\big(T(x_n, y_n, z_n, t_n), T(u, v, w, s)\big) + d_b(x_{n+1}, u)\big] \\ \preceq & ba_1 d_b\big(T(x_n, y_n, z_n, t_n), u\big) + ba_2 d_b\big(T(u, v, w, s), x_n\big) + bd_b(x_{n+1}, u) \\ \preceq & ba_1 d_b(x_{n+1}, u) + ba_2 d_b\big(T(u, v, w, s), x_n\big) + bd_b(x_{n+1}, u). \end{aligned} \tag{2.33}$$

Thus

$$\begin{aligned} & \big\|d_b\big(T(u, v, w, s), u\big)\big\| \\ & \leq \|ba_1\| \big\|d_b(x_{n+1}, u)\big\| + \|ba_2\| \big\|d_b\big(T(u, v, w, s), x_n\big)\big\| + \|b\| \big\|d_b(x_{n+1}, u)\big\| \\ & \to \|ba_2\| \big\|d_b\big(T(u, v, w, s), u\big)\big\|, \quad n \to \infty. \end{aligned} \tag{2.34}$$

Since $0_{\mathbb{A}} \preceq ba_2 \preceq (a_1 + a_2)b$, we have $\|a_2 b\| \leq \|(a_1 + a_2)b\| < 1$ by Lemma 7(6). This and 2.34 imply that $\|d_b(T(u, v, w, s), u)\| = 0$. Hence $T(u, v, w, s) = u$. Similarly, we obtain $T(v, w, s, u) = v$, $T(w, s, u, v) = w$ and $T(s, u, v, w) = s$. Thus (u, v, w, s) is a quadrupled fixed point of T. Now if

(u^*, v^*, w^*, s^*) is another quadrupled fixed point of T, then

$$\begin{aligned} 0_{\mathbb{A}} \preceq & d_b\big(u, u^*\big) = d_b\big(T(u, v, w, s), T\big(u^*, v^*, w^*, s^*\big)\big) \\ \preceq & a_1 d_b\big(T(u, v, w, s), u^*\big) + a_2 d_b\big(T\big(u^*, v^*, w^*, s^*\big), u\big) \\ \preceq & a_1 d_b\big(u, u^*\big) + a_2 d_b\big(u^*, u\big) = (a_1 + a_2) d_b\big(u, u^*\big), \end{aligned} \tag{2.35}$$

so, we get

$$0 \leq \big\|d_b\big(u, u^*\big)\big\| \leq \|a_1 + a_2\| \big\|d_b\big(u, u^*\big)\big\| < \frac{1}{\|b\|^2} \big\|d_b\big(u, u^*\big)\big\| \leq \big\|d_b(u, v)\big\|, \tag{2.36}$$

which implies that $\|d_b(u, u^*)\| = 0$, then we have $u = u^*$. Similarly, we can get $v = v^*$, $w = w^*$ and $s = s^*$. Hence, the quadrupled fixed point is unique. Moreover, we will prove the uniqueness of fixed points of T. By 2.21, we have

$$\begin{aligned} d_b(u, v) = & d_b\big(T(u, v, w, s), T(v, w, s, u)\big) \\ \preceq & a_1 d_b\big(T(u, v, w, s), v\big) + a_2 d_b\big(T(v, w, s, u), u\big) = (a_1 + a_2) d_b(u, v), \end{aligned} \tag{2.37}$$

similarly we can show that

$$\begin{aligned} d_b(v, w) = & d_b\big(T(v, w, s, u), T(w, s, u, v)\big) \\ \preceq & a_1 d_b\big(T(v, w, s, u), w\big) + a_2 d_b\big(T(w, s, u, v), v\big) = (a_1 + a_2) d_b(v, w), \end{aligned} \tag{2.38}$$

$$\begin{aligned} d_b(w, s) = & d_b\big(T(w, s, u, v), T(s, u, v, w)\big) \\ \preceq & a_1 d_b\big(T(w, s, u, v), s\big) + a_2 d_b\big(T(s, u, v, w), w\big) = (a_1 + a_2) d_b(w, s), \end{aligned} \tag{2.39}$$

$$\begin{aligned} d_b(s, u) = & d_b\big(T(s, u, v, w), T(u, v, w, s)\big) \\ \preceq & a_1 d_b\big(T(s, u, v, w), u\big) + a_2 d_b\big(T(u, v, w, s), s\big) = (a_1 + a_2) d_b(s, u), \end{aligned} \tag{2.40}$$

then

$$\big\|d_b\big(u, u^*\big)\big\| \leq \|a_1 + a_2\| \big\|d_b\big(u, u^*\big)\big\| < \frac{1}{\|b\|^2} \big\|d_b\big(u, u^*\big)\big\| \leq \big\|d_b(u, v)\big\|, \tag{2.41}$$

which yields $u = v = w = s$. This completes the proof. □

Theorem 13. *Let (X, A, d) be a complete C^*-algebra-valued metric space. Suppose that the mapping $F : X^4 \to X$ satisfies the following condition*

$$d(F(x, y, z), F(u, v, w)) \preceq ad(F(x, y, z, t), u) + bd(F(u, v, w, s), x),$$

for any $x, y, z, t, u, v, w, s \in X$, where $a, b \in A'_+$ with $\|a\| + \|b\| < 1$. Then F has a unique quadrupled fixed point. Moreover, F has a unique fixed point in X.

Proof. Taking $b = 1_{\mathbb{A}}$, becomes a special case of Theorem 12. □

Theorem 14. *Let $(X, \mathbb{A}, d_b)$ be a complete C^* -valued b-metric space. Assume that the mapping $T : X^4 \to X$ satisfies the following condition:*

$$d_b\big(T(x,y,z,t),T(u,v,w,s)\big) \preceq a_1 d_b\big(T(x,y,z,t),x\big) + a_2 d_b\big(T(u,v,w,s),u\big), \tag{2.42}$$

where $x, y, z, t, u, v, w, s \in X$ and $a_1, a_2 \in \mathbb{A}'_+$ with $(\|a_1\| + \|a_2\|)\|b\| < 1$. Then T has a unique quadrupled fixed point in X. Moreover, T has a unique fixed point.

Proof. Since $a_1, a_2 \in \mathbb{A}'_+$, we see that $a_1 d_b\big(T(x,y,z,t),x\big) + a_2 d_b\big(T(u,v,w,s),u\big)$ is a positive element. Similar to the proof of Theorem 12, we construct $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ such that $x_{n+1} = T(x_n, y_n, z_n, t_n)$, $y_{n+1} = T(y_n, z_n, t_n, x_n)$, $z_{n+1} = T(z_n, t_n, x_n, y_n)$ and $t_{n+1} = T(t_n, x_n, y_n, z_n)$. By 2.42, we obtain

$$\begin{aligned} d_b(x_n, x_{n+1}) =& d_b\big(T(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), T(x_n, y_n, z_n, t_n)\big) \\ \preceq& a_1 d_b\big(T(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), x_{n-1}\big) + a_2 d_b\big(T(x_n, y_n, z_n, t_n), x_n\big) \\ =& a_1 d_b(x_n, x_{n-1}) + a_2 d_b(x_{n+1}, x_n), \end{aligned} \tag{2.43}$$

which implies that

$$(1_{\mathbb{A}} - a_2) d_b(x_n, x_{n+1}) \preceq a_1 d_b(x_n, x_{n-1}).$$

Since $a_1, a_2 \in \mathbb{A}'_+$ with $\|a_1\| + \|a_2\| < \frac{1}{\|b\|} \leq 1$, we have $1_{\mathbb{A}} - a_2$ is invertible and $(1_{\mathbb{A}} - a_2)^{-1} a_1 \in \mathbb{A}'_+$. Hence

$$d_b(x_n, x_{n+1}) \preceq (1_{\mathbb{A}} - a_2)^{-1} a_1 d_b(x_n, x_{n-1}).$$

Inductively, for all $n \in \mathbb{N}$, we have

$$d_b(x_n, x_{n+1}) \preceq k^n m_0, \tag{2.44}$$

where $k = (1_{\mathbb{A}} - a_2)^{-1} a_1$ and $m_0 = d_b(x_1, x_0)$. Since $\|a_1\|\|b\| + \|a_2\| \leq (\|a_1\| + \|a_2\|)\|b\| < 1$, we have

$$\|bk\| = \big\|(1_{\mathbb{A}} - a_2)^{-1} a_1 b\big\| \leq \big\|(1_{\mathbb{A}} - a_2)^{-1}\big\| \|a_1\|\|b\| = \sum_{i=0}^{\infty} \|a_2\|^i \|a_1\|\|b\| = \frac{\|a_1\|\|b\|}{1 - \|a_2\|} < 1. \tag{2.45}$$

And $\|k\| \leq \|bk\| < 1$ by Lemma 7(6). Let $n, m \in \mathbb{N}$ with $m > n$, by using Definition 1.1, (2.16), and (2.17), we have

$$\begin{aligned} d_b(x_n, x_m) \preceq & b\big[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_m)\big] \\ \preceq & bd_b(x_n, x_{n+1}) + b^2 d_b(x_{n+1}, x_{n+2}) + \cdots \\ & + b^{m-n-1}\big[d_b(x_{m-2}, x_{m-1}) + d_b(x_{m-1}, x_m)\big] \\ \preceq & bk^n M_0 + b^2 k^{n+1} M_0 + \cdots + b^{m-n-1} k^{m-2} M_0 + b^{m-n-1} k^{m-1} M_0 \\ = & \sum_{i=1}^{m-n-1} b^i k^{n+i-1} M_0 + b^{m-n-1} k^{m-1} M_0 \\ = & \sum_{i=1}^{m-n-1} \big|M_0^{\frac{1}{2}} k^{\frac{n+i-1}{2}} b^{\frac{i}{2}}\big|^2 + \big|M_0^{\frac{1}{2}} k^{\frac{m-1}{2}} b^{\frac{m-n-1}{2}}\big|^2 \\ \preceq & \sum_{i=1}^{m-n-1} \big\|M_0^{\frac{1}{2}} k^{\frac{n+i-1}{2}} b^{\frac{i}{2}}\big\|^2 1_{\mathbb{A}} + \big\|M_0^{\frac{1}{2}} k^{\frac{m-1}{2}} b^{\frac{m-n-1}{2}}\big\|^2 1_{\mathbb{A}} \\ \preceq & \|M_0\| \sum_{i=1}^{m-n-1} \big\|(bk)^{\frac{i}{2}}\big\|^2 \big\|k^{\frac{n-1}{2}}\big\|^2 1_{\mathbb{A}} + \|M_0\| \big\|(bk)^{\frac{m-n-1}{2}}\big\|^2 \big\|k^{\frac{n}{2}}\big\|^2 1_{\mathbb{A}} \\ = & \|M_0\| \|k\|^{n-1} \sum_{i=1}^{m-n-1} \|bk\|^i 1_{\mathbb{A}} + \|M_0\| \|bk\|^{m-n-1} \|k\|^n 1_{\mathbb{A}} \\ = & \|M_0\| \|k\|^{n-1} \frac{\|bk\| - \|bk\|^{m-n}}{1 - \|bk\|} 1_{\mathbb{A}} + \|M_0\| \|bk\|^{m-n-1} \|k\|^n 1_{\mathbb{A}} \\ \preceq & \frac{\|M_0\| \|bk\|}{1 - \|bk\|} \|k\|^{n-1} 1_{\mathbb{A}} + \|M_0\| \|bk\|^{m-n-1} \|k\|^n 1_{\mathbb{A}} \\ \to & 0_{\mathbb{A}} \quad (\text{as } m, n \to \infty). \end{aligned} \tag{2.46}$$

Hence $\{x_n\}$ is a Cauchy sequence. Similarly, we can prove that $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are also a Cauchy sequence. Since $(X, \mathbb{A}, d_b)$ is complete, there are $u, v, w, s \in X$ such that $x_n \to u$, $y_n \to v$, $z_n \to w$ and $t_n \to s$ as $n \to \infty$. In the following, we will show that $T(u, v, w, s) = u$, $T(v, w, s, u) = v$, $T(w, s, u, v) = w$ and $T(s, u, v, w) = s$. From 2.42, we get

$$\begin{aligned} d_b\big(T(u, v, w, s), u\big) \preceq & b\big[d_b\big(x_{n+1}, T(u, v, w, s)\big) + d_b(x_{n+1}, u)\big] \\ \preceq & b\big[d_b\big(T(x_n, y_n, z_n, t_n), T(u, v, w)\big) + d_b(x_{n+1}, u)\big] \\ \preceq & b\big[a_1 d_b\big(T(x_n, y_n, z_n, t_n), x_n\big) + a_2 d_b\big(T(u, v, w, s), u\big) + d_b(x_{n+1}, u)\big] \\ = & ba_1 d_b(x_{n+1}, x_n) + ba_2 d_b\big(T(u, v, w, s), u\big) + bd_b(x_{n+1}, u), \end{aligned} \tag{2.47}$$

which implies that

$$d_b\big(T(u, v, w, s), u\big) \preceq (1_{\mathbb{A}} - a_2 b)^{-1} ba_1 d_b(x_{n+1}, x_n) + (1_{\mathbb{A}} - a_2 b)^{-1} d_b(x_{n+1}, u). \tag{2.48}$$

Thus $d_b(T(u,v,w,s),u) = 0_{\mathbb{A}}$. Equivalently, $T(u,v,w,s) = u$. Similarly, we can obtain $T(v,w,s,u) = v$, $T(w,s,u,v) = w$ and $T(s,u,v,w) = s$. Now if (u^*,v^*,w^*,s^*) is another quadrupled fixed point of T, then

$$\begin{aligned} 0_{\mathbb{A}} \preceq & d_b\big(u,u^*\big) = d_b\big(T(u,v,w,s),T\big(u^*,v^*,w^*,s^*\big)\big) \\ \preceq & a_1 d_b\big(T(u,v,w,s),u\big) + a_2 d_b\big(T\big(u^*,v^*,w^*,s^*\big),u^*\big) = a_1 d_b(u,u) + a_2 d_b\big(u^*,u^*\big) = 0_{\mathbb{A}}, \end{aligned} \tag{2.49}$$

so, we get $d_b(u,u^*) = 0_{\mathbb{A}}$, which yields $u^* = u$. Similarly, we have $v^* = v$, $w^* = w$ and $s^* = s$. Thus, (u,v,w,s) is the unique quadrupled fixed point of T. Finally, we will show the uniqueness of fixed points of T. By 2.42, we have

$$\begin{aligned} d_b(u,v) = & d_b\big(T(u,v,w,s),T(v,w,s,u)\big) \\ \preceq & a_1 d_b\big(T(u,v,w,s),u\big) + a_2 d_b\big(T(v,w,s,u),v\big) = 0_{\mathbb{A}}, \end{aligned} \tag{2.50}$$

similarly we can show that

$$\begin{aligned} d_b(v,w) = & d_b\big(T(v,w,s,u),T(w,s,u,v)\big) \\ \preceq & a_1 d_b\big(T(v,w,s,u),v\big) + a_2 d_b\big(T(w,s,u,v),w\big) = 0_{\mathbb{A}}, \end{aligned} \tag{2.51}$$

$$\begin{aligned} d_b(w,s) = & d_b\big(T(w,s,u,v),T(s,u,v,w)\big) \\ \preceq & a_1 d_b\big(T(w,s,u,v),w\big) + a_2 d_b\big(T(s,u,v,w),s\big) = 0_{\mathbb{A}}, \end{aligned} \tag{2.52}$$

$$\begin{aligned} d_b(s,u) = & d_b\big(T(s,u,v,w),T(u,v,w,s)\big) \\ \preceq & a_1 d_b\big(T(s,u,v,w),s\big) + a_2 d_b\big(T(u,v,w,s),u\big) = 0_{\mathbb{A}}, \end{aligned} \tag{2.53}$$

which implies that $u = v = w = s$.

□

Theorem 15. *Let (X,A,d) be a complete C^*-algebra-valued metric space. Suppose that the mapping $F : X^4 \to X$ satisfies the following condition*

$$d(F(x,y,z,t),F(u,v,w,s)) \preceq ad(F(x,y,z,t),x) + bd(F(u,v,w,s),u),$$

for any $x,y,z,t,u,v,w,s \in X$, where $a,b \in A_+$ with $\|a\|+\|b\| < 1$. Then F has a unique quadrupled fixed point. Moreover, F has a unique fixed point in X.

Proof. Taking $b = 1_{\mathbb{A}}$, becomes a special case of Theorem 14. □

3. Application

As an application of quadrupled fixed point theorems on complete C^* -algebra-valued b-metric spaces, we prove here the existence and uniqueness of a solution for a Fredholm nonlinear integral equation. Let E be a Lebesgue-measurable set with $m(E) < \infty$ and $X = L^\infty(E)$ denote the class of essentially bounded measurable functions on E. Consider the Hilbert space $L^2(E)$. Let the set of all bounded linear operators on $L^2(E)$ be denoted by $B(L^2(E))$. Obviously, $B(L^2(E))$ is a C^*-algebra with usual operator norm. Let $K_1, K_2, K_3, K_4 : E^4 \to \mathbb{R}$, assume that there exist four continuous functions $f_1, f_2, f_3, f_4 : E^4 \to \mathbb{R}$ and a constant $\alpha \in \left(0, \frac{1}{16}\right)$ such that for all $x, y \in X$ and $u, v, w, s \in E$, we have

$$\left|K_1\big(u,v,w,s,x(v)\big) - K_1\big(u,v,w,s,y(v)\big)\right| \le \alpha\left|f_1(u,v,w,s)\right|\left|x(v)-y(v)\right|, \tag{3.1}$$

$$\left|K_2\big(u,v,w,s,x(v)\big) - K_2\big(u,v,w,s,y(v)\big)\right| \le \alpha\left|f_2(u,v,w,s)\right|\left|x(v)-y(v)\right|, \tag{3.2}$$

$$\left|K_3\big(u,v,w,s,x(v)\big) - K_3\big(u,v,w,s,y(v)\big)\right| \le \alpha\left|f_3(u,v,w,s)\right|\left|x(v)-y(v)\right|, \tag{3.3}$$

$$\left|K_4\big(u,v,w,s,x(v)\big) - K_4\big(u,v,w,s,y(v)\big)\right| \le \alpha\left|f_4(u,v,w,s)\right|\left|x(v)-y(v)\right|, \tag{3.4}$$

Example 16. *Consider the integral equation*

$$x(t) = \int_E \Big(K_1\big(u,v,w,s,x(v)\big) + K_2\big(u,v,w,s,x(v)\big) + K_3\big(u,v,w,s,x(v)\big) + K_4\big(u,v,w,s,x(v)\big)\Big)\, dv, \tag{3.5}$$

where $u, v, w, s \in E$.

Assume that 3.1, 3.2, 3.3 and 3.4 hold. Moreover, if

$$\begin{aligned} &\sup_{u\in E}\int_E \left|f_1(u,v,w,s)\right| dv \le 1, \qquad \sup_{v\in E}\int_E \left|f_2(u,v,w,s)\right| dv \le 1,\\ &\sup_{w\in E}\int_E \left|f_3(u,v,w,s)\right| dv \le 1, \qquad \sup_{s\in E}\int_E \left|f_4(u,v,w,s)\right| dv \le 1 \end{aligned} \tag{3.6}$$

then the integral equation 3.5 has a unique solution in $L^\infty(E)$.

Proof. Define $d_b : X \times X \to B(L^2(E))$ as follows:

$$d_b(f,g) = \pi_{(f-g)^2},$$

where $\pi_h : L^2(E) \to L^2(E)$ is the product operator given by

$$\pi_h(u) = h \cdot u \quad \text{for } u \in L^2(E).$$

Working in the same lines as in [15], Example 3.2, we easily see that $(X, B(L^2(E)), d_b)$ is a complete C^* -valued b-metric space with $b = 2 \cdot 1_{B(L^2(E))}$. Let $T : X^4 \to X$ be

$$T(x_1,x_2,x_3,x_4)(t) = \int_E \Big(K_1\big(t,s,r,w,x_1(s)\big) + K_2\big(t,s,r,w,x_2(s)\big) + K_3\big(t,s,r,w,x_3(s)\big) + K_4\big(t,s,r,w,x_4(s)\big)\Big)\, ds,$$

where $t, s, r, w \in E$. Then by 3.1, 3.2, 3.3, 3.4 and 3.6, we obtain

$$\begin{aligned}
&\left\|d_b\big(T(x_1,x_2,x_3,x_4),T(u_1,u_2,u_3,u_4)\big)\right\| \\
&= \sup_{\|p\|=1} \langle \pi_{|T(x,y,z)-T(u,v,w)|}p,p\rangle \quad \text{for every } p \in L^2(E) \\
&= \sup_{\|p\|=1} \int_E \left|T(x_1,x_2,x_3,x_4)-T(u_1,u_2,u_3,u_4)\right|^2 p(t)\overline{p(t)}\,dt \\
&\le 4 \sup_{\|p\|=1} \int_E \left[\int_E \left|K_1\big(t,s,r,w,x_1(s)\big)-K_1\big(t,s,r,w,u_1(s)\big)\right| ds\right]^2 \left|p(t)\right|^2 dt \\
&\quad + 4 \sup_{\|p\|=1} \int_E \left[\int_E \left|K_2\big(t,s,r,w,x_2(s)\big)-K_2\big(t,s,r,w,u_2(s)\big)\right| ds\right]^2 \left|p(t)\right|^2 dt \\
&\quad + 4 \sup_{\|p\|=1} \int_E \left[\int_E \left|K_3\big(t,s,r,w,x_3(s)\big)-K_3\big(t,s,r,w,u_3(s)\big)\right| ds\right]^2 \left|p(t)\right|^2 dt \\
&\quad + 4 \sup_{\|p\|=1} \int_E \left[\int_E \left|K_4\big(t,s,r,w,x_4(s)\big)-K_4\big(t,s,r,w,u_4(s)\big)\right| ds\right]^2 \left|p(t)\right|^2 dt
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\le 4 \sup_{\|p\|=1} \int_E \alpha^2 \left[\int_E \left|f_1(t,s,r,w)\right|\left|x_1(s)-u_1(s)\right| ds\right]^2 \left|p(t)\right|^2 dt \\
&\quad + 4 \sup_{\|p\|=1} \int_E \alpha^2 \left[\int_E \left|f_2(t,s,r,w)\right|\left|x_2(s)-u_2(s)\right| ds\right]^2 \left|p(t)\right|^2 dt \\
&\quad + 4 \sup_{\|p\|=1} \int_E \alpha^2 \left[\int_E \left|f_3(t,s,r,w)\right|\left|x_3(s)-u_3(s)\right| ds\right]^2 \left|p(t)\right|^2 dt \\
&\quad + 4 \sup_{\|p\|=1} \int_E \alpha^2 \left[\int_E \left|f_4(t,s,r,w)\right|\left|x_4(s)-u_4(s)\right| ds\right]^2 \left|p(t)\right|^2 dt
\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
\leq & 4\alpha^2 \sup_{\|p\|=1} \int_E \left[\int_E \left| f_1(t,s,r,w) \right| ds \right]^2 \left| p(t) \right|^2 dt \cdot \left\| (x_1 - u_1)^2 \right\|_\infty \\
& + 4\alpha^2 \sup_{\|p\|=1} \int_E \left[\int_E \left| f_2(t,s,r,w) \right| ds \right]^2 \left| p(t) \right|^2 dt \cdot \left\| (x_2 - u_2)^2 \right\|_\infty \\
& + 4\alpha^2 \sup_{\|p\|=1} \int_E \left[\int_E \left| f_3(t,s,r,w) \right| ds \right]^2 \left| p(t) \right|^2 dt \cdot \left\| (x_3 - u_3)^2 \right\|_\infty \\
& + 4\alpha^2 \sup_{\|p\|=1} \int_E \left[\int_E \left| f_4(t,s,r,w) \right| ds \right]^2 \left| p(t) \right|^2 dt \cdot \left\| (x_4 - u_4)^2 \right\|_\infty \\
\leq & 4\alpha^2 \sup_{t \in E} \left[\int_E \left| f(t,s,r,w) \right| ds \right]^2 \cdot \sup_{\|p\|=1} \int_E \left| p(t) \right|^2 dt \cdot \left\| (x_1 - u_1)^2 \right\|_\infty \qquad (3.9) \\
& + 4\alpha^2 \sup_{t \in E} \left[\int_E \left| f_2(t,s,r,w) \right| ds \right]^2 \cdot \sup_{\|p\|=1} \int_E \left| p(t) \right|^2 dt \cdot \left\| (x_2 - u_2)^2 \right\|_\infty \\
& + 4\alpha^2 \sup_{t \in E} \left[\int_E \left| f_3(t,s,r,w) \right| ds \right]^2 \cdot \sup_{\|p\|=1} \int_E \left| p(t) \right|^2 dt \cdot \left\| (x_3 - u_3)^2 \right\|_\infty \\
& + 4\alpha^2 \sup_{t \in E} \left[\int_E \left| f_4(t,s,r,w) \right| ds \right]^2 \cdot \sup_{\|p\|=1} \int_E \left| p(t) \right|^2 dt \cdot \left\| (x_4 - u_4)^2 \right\|_\infty \\
\leq & 4\alpha^2 \left(\left\| (x_1 - u_1)^2 \right\|_\infty + \left\| (x_2 - u_2)^2 \right\|_\infty + \left\| (x_3 - u_3)^2 \right\|_\infty + \left\| (x_4 - u_4)^2 \right\|_\infty \right) \\
= & 4\alpha^2 d_b(x_1, u_1) + 4\alpha^2 d_b(x_2, u_2) + 4\alpha^2 d_b(x_3, u_3) + 4\alpha^2 d_b(x_4, u_4).
\end{aligned}
$$

Set $a = 2\alpha 1_{B(L^2(E))}$, then $a \in B(L^2(E))$ and $\|a\| = 2\alpha < \frac{1}{8} = \frac{1}{2\|b\|}$. Hence, all the conditions of Theorem 10 hold. Applying Theorem 10, we see that the integral equation 3.5 has a unique solution in $L^\infty(E)$.

□

Example 17. *Suppose that H is a Hilbert space, $L(H)$ is the set of linear bounded operators on H. Let $A_1, A_2, \ldots, A_n, \ldots, B_1, B_2, \ldots, B_n, \ldots C_1, C_2, \ldots, C_n, \ldots D_1, D_2, \ldots, D_n, \cdots \in L(H)$ which satisfy $\Sigma_{n=1}^\infty (\|A_n\| + \|B_n\| + \|C_n\| + \|D_n\|) \leq 1$ and $Q \in L(H)_+$. Then the operator equation*

$$X - \Sigma_{n=1}^\infty (A_n^* X A_n + B_n^* X B_n + C_n^* X C_n + D_n^* X D_n) = Q$$

has a unique solution in $L(H)$.

Proof. Set $\alpha = (\Sigma_{n=1}^\infty \|A_n + B_n + C_n + D_n\|)^p$ with $p \geq 1$, then $\|\alpha\| \leq 1$. Without loss of generality, one can suppose that $\alpha > 0$. Choose a positive operator $T \in L(H)$. For $X_1, X_2, X_3, X_4 \in L(H)$ and $p \geq 1$, set

$$d(X, Y) = \|X - Y\|^p T.$$

Then $d(X, Y)$ is a C^*-algebra-valued b-metric and $(L(H), d)$ is complete since $L(H)$ is a Banach space. Indeed, it suffices to check the third condition of Definition 4 as follows. Suppose that $X, Y, Z \in L(H)$ and set $U = X - Z, V = Z - Y$. Using the well-known inequality $(a + b)^p \leq$

$(2\max\{a,b\})^p \leq 2^p(a^p+b^p)$ for all $a,b \geq 0$, we have

$$\begin{aligned}\|X-Y\|^p &= \|U+V\|p \leq (\|U\|+\|V\|)^p \\ &\leq 2^p(\|U\|^p+\|V\|^p) \\ &= 2^p(\|X-Z\|^p+\|Z-Y\|p),\end{aligned}$$

which implies that

$$d(X,Y) \leq A[d(X,Z)+d(Z,Y)],$$

where $A = 2^p I$. Consider the map $F : (L(H))^4 \to L(H)$ defined by

$$F(X_1,X_2,X_3,X_4) = \Sigma_{n=1}^{\infty}(\|A_n\|+\|B_n\|+\|C_n\|+\|D_n\|)+Q$$

Then

$$\begin{aligned}&d(F(X_1,X_2,X_3,X_4),F(U_1,U_2,U_3,U_4)) \\ =\ & \|F(X_1,X_2,X_3,X_4)-F(U_1,U_2,U_3,U_4)\|^p T \\ =\ & \|\Sigma_{n=1}^{\infty}(A_n^*(X_1-U_1)A_n+B_n^*(X_2-U_2)B_n+C_n^*(X_3-U_3)C_n)+D_n^*(X_4-U_4)C_n)\|^p T \\ \leq\ & \|\Sigma_{n=1}^{\infty}A_n^*(X_1-U_1)A_n\|^p+\|\Sigma_{n=1}^{\infty}B_n^*(X_2-U_2)B_n\|^p \\ & +\|\Sigma_{n=1}^{\infty}C_n^*(X_3-U_3)C_n\|^p T+\|\Sigma_{n=1}^{\infty}D_n^*(X_4-U_4)C_n\|^p T \\ \leq\ & (\Sigma_{n=1}^{\infty}\|A_n\|^{2p}\|X_1-U_1\|^p+\Sigma_{n=1}^{\infty}\|B_n\|^{2p}\|X_2-U_2\|^p \\ & +\Sigma_{n=1}^{\infty}\|C_n\|^{2p}\|X_3-U_3\|^p+\Sigma_{n=1}^{\infty}\|D_n\|^{2p}\|X_4-U_4\|^p \\ \leq\ & \alpha^2(d(X_1,U_1)+d(X_2,U_2)+d(X_3,U_3)+d(X_4,U_4)) \\ =\ & (\alpha I)^*(d(X_1,U_1)+d(X_2,U_2)+d(X_3,U_3)+d(X_4,U_4))(\alpha I).\end{aligned}$$

Using Theorem 10, there exists a unique quadrupled fixed point X in $L(H)$. Furthermore, since $\Sigma_{n=1}^{\infty}(A_n^*XA_n+B_n^*XB_n+C_n^*XC_n+D_n^*XD_n)+Q$ is a positive operator, the solution is a Hermitian operator.

□

4. Acknowledgements

References

[1] R. P. Agarwal, M. Meehan, and D. ORegan: *Fixed Point Theory and Application.* Cambridge University Press, Cambridge (2001).

[2] M A, Alghamdi, N, Hussain, P Salimi,: *Fixed point and coupled fixed point theorems on b-metric-like spaces.* J. Inequal. Appl. 2013, **402** (2013)

[3] M S, Asgari, B Mousavi,: *Solving a class of nonlinear matrix equations via the coupled fixed point theorem.* Appl. Math. Comput. **259**(2015), 364-373

[4] I A Bakhtin, : *The contraction principle in quasimetric spaces. In: Functional Analysis,* vol. **30,**(1989) pp. 26-37

[5] Lj. B. Ćirić : *Multi-valued nonlinear contraction mappings.* Nonlinear Anal. **71**(2009), pp 2716–2723 . doi:10.1016/j. na.2009.01.116

[6] Lj. B. Ćirić : *Fixed point theorems for multi-valued contractions in complete metric spaces.* J Math Anal Appl. **348**(2008), pp 499–507 . doi:10.1016/j.jmaa.2008.07.062

[7] Hamidreza Rahimi, Stojan Radenović, Chasem Soleimani Rad and Poom Kumam: *Quadrupled fixed point results in abstract metric spaces,* Computational and Applied Mathematics; 2013, DOI 10.1007/s40314-013-0088-5.

[8] H, Huang, S Radenovic,: *Common fixed point theorems of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras and applications.* J. Nonlinear Sci. Appl. **8(5)**(2015) , 787-799

[9] Kamran, T, Postolache, M, Ghiura, A, Batul, S, Ali, R: *The Banach contraction principle in C^*-algebra-valued b-metric spaces with application.* Fixed Point Theory Appl. 2016, **10** (2016)

[10] Z Kadelburg, , S Radenovic, : *Pata-type common fixed point results in b-metric and b-rectangular metric spaces.* J. Nonlinear Sci. Appl. **8(6)**(2015), 944-954

[11] E. Karapiner , *Quadruple Fixed Point Theorems for Weak ϕ- Contraction,* ISRN Math. Anal. (2011), ID 989423, 15 pages, doi:10.5402/2011/989423.

[12] A Latif, Z, Kadelburg, V, Parvaneh, JR Roshan,: *Some fixed point theorems for G-rational Geraghty contractive mappings in ordered generalized b-metric spaces.* J. Nonlinear Sci. Appl. **8(6)**, 1212-1227 (2015)

[13] Murphy, GJ : C^*-Algebras and Operator Theory. Academic Press, London (1990)

[14] NV, Luong, NX Thuan, : Coupled fixed points in partially ordered metric spaces and application. Nonlinear Anal. **74**(2011), 983-992

[15] Z, Ma, L Jiang : C^*-Algebra-valued b-metric spaces and related fixed point theorems. Fixed Point Theory Appl. 2015, **222** (2015)

[16] J. J. Nieto and R. Rodriguez - Lopez: *Fixed point theorems in ordered abstract spaces.* Proc Am Math Soc. **135** (2007), pp 2505–2517. doi:10.1090/S0002-9939-07-08729-1

[17] J. J. Nieto and R. Rodriguez - Lopez: *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation.* Order **22**(2005), pp 223–239.

[18] J. J. Nieto and R. Rodriguez - Lopez: *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations.* Acta Math. Sin. Engl. Ser. **23(12)**(2007), pp 2205–2212.

[19] D. Paesano and P. Vetro, : *Suzukis type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces.* Topol. Appl. **159(3)** (2012), pp 911–920

[20] A C M, Ran and M C B Reurings: *A fixed point theorem in partially ordered sets and some applications to matrix equations.* Proc. Am.Math. Soc. **132**(2004), pp 1435–1443.

[21] B. Samet and C. Vetro, : *Coupled fixed point theorems for multi-valued bilinear contraction mappings in partially ordered metric spaces.* Nonlinear Anal. **74**(2011), pp 4260–4268 . doi:10.1016/j.na.2011.04.007

[22] MH, Shah, N Hussain,: *Nonlinear contractions in partially ordered quasi b-metric spaces.* Commun. Korean Math. Soc.**27(1)**(2012), 117-128

[23] CX, Zhu, CF, Chen, X Zhang,: *Some results in quasi-b-metric-like spaces.* J. Inequal. Appl. 2014, **437** (2014)

(Research Supervisor), Govt. Dr. Shyama Prashad Mukherjee Science & Commerce College, (Old Benajir) Bhopal (M.P.).

(Research Scholar), Govt. Dr. Shyama Prashad Mukherjee Science & Commerce College, (Old Benajir) Bhopal (M.P.).

E-mail address: sachinmokhle08@gmail.com